

Constant rotation of two-qubit equally entangled pure states by local quantum operations

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Abstract

We look for local unitary operators $W_1 \otimes W_2$ which would rotate all equally entangled two-qubit pure states by the same but arbitrary amount. It is shown that all two-qubit maximally entangled states can be rotated through the same but arbitrary amount by local unitary operators. But there is no local unitary operator which can rotate all equally entangled non-maximally entangled states by the same amount, unless it is unity. We have found the optimal sets of equally entangled non-maximally entangled states which can be rotated by the same but arbitrary amount via local unitary operators $W_1 \otimes W_2$, where at most one these two operators can be identity. In particular, when $W_1 = W_2 = (i/\sqrt{2})(\sigma_x + \sigma_y)$, we get the local quantum NOT operation. Interestingly, when we apply the one-sided local depolarizing map, we can rotate all equally entangled two-qubit pure states through the same amount. We extend our result for the case of three-qubit maximally entangled state.

1 Introduction

Fundamental limitation of certain quantum operations of single systems with regards to global operations, has already been studied in the literature [1, 2, 3, 4]. Cloning and the

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NOT operation, applied to the qubit, are the two prime example [1, 3]. It is well known that an arbitrary state, taken from a set of two known non-orthogonal states, can not be copied exactly. But any state taken from a set of orthogonal states can be copied exactly. Similarly, there exists no universal flipper which can operate on a unknown qubit state $|\psi\rangle$, resulting in the orthogonal state $|\psi^\perp\rangle$. The largest set of states (of single-qubit system), which can be flipped exactly by a single unitary operator, is the set of states lying on a great circle of the Bloch sphere [4].

Entanglement lies at the heart of many aspects of quantum information theory and it is therefore desirable to understand its structure as well as possible [5]. One attempt to improve our understanding of entanglement is the study of our ability to perform information theoretic tasks locally on entangled state, such as local discrimination, local cloning etc. [6, 7, 8, 9]. It has already been established that these local scenarios of quantum operations are different from the global scenarios. For example, in the global scenario any state given from the set of orthogonal states can be cloned exactly, whereas there are examples of set of orthogonal entangled states which can not be cloned exactly by local operation [8, 9].

Recently Novotný et al. [10] studied the optimal covariant⁵ quantum NOT operations for equally entangled qubit pairs, in the global scenario. In particular, they have shown that only in the case of maximally entangled input states, such covariant quantum NOT operations can be performed perfectly. In the case of maximally entangled states of two qubits, they have also discussed about universal non-covariant quantum NOT operations. Motivated by these studies, and in the line of the tasks of local cloning, local deleting, local broadcasting, locally distinguishing, etc., we would like to investigate here the possibility of locally rotating, through the same amount, equally entangled two-qubit states. We here consider implementation of an exact quantum NOT operation, and its generalization (namely, rotation) by using LOCC only. In particular, we will show here that *all* two-qubit maximally entangled states can be exactly rotated through the same (but arbitrarily given) amount by the action of an one-sided local unitary operator. We also show that for given an arbitrary value of two-qubit pure entanglement (E , say), there exists a maximal subset of the set of all two-qubit pure states having entanglement E , such that all the elements of that maximal set can be exactly rotated through the same (but arbitrarily given) amount by the action of an one-sided local unitary operator⁶. It is shown that in the case of equally entangled non-maximally entangled two-qubit pure states, not all the states in the whole class of such states can be rotated through a constant amount by local unitary operation. But in the case of maximally entangled states, all the states in this class can be rotated through a constant amount by local unitary operation. As a consequence of this general studies we will discuss the local-NOT operation of pure

⁵If a completely positive map $T : \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2) \rightarrow \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2)$ has to treat pure two-qubit density matrices ρ , of given degree of entanglement, in a *covariant* way, one must have $T((U \otimes V)\rho(U^\dagger \otimes V^\dagger)) = (U \otimes V)T(\rho)(U^\dagger \otimes V^\dagger)$ for all $U, V \in SU(2)$.

⁶Throughout this paper, we take only those unitary matrices each of whose determinant is unity. But the results of this paper also follows for unitary operators having determinant other than unity – simply we need to multiply the respective special unitary operators by a phase.

two-qubit states with a fixed degree of entanglement.

In section 2, we describe the general scheme of ‘rotating’, via same amount, pure states of $2 \otimes 2$ having same amount of entanglement, by local unitary operators $W_1 \otimes W_2$, where $W_1, W_2 \in SU(2)$. In section 3, we consider the issue of rotation, through the same amount, of equally entangled two-qubit pure states by one-sided unital trace-preserving completely (CP) positive maps. In section 4, we consider the issue of rotating, via same amount, maximally entangled states of three qubits by local unitary operators. We shall draw the conclusion in section 5.

2 Constant rotation of equally entangled two-qubit pure states by local unitary operators

To start with, we consider a two-qubit pure state

$$|\psi_0\rangle = l_0|00\rangle + l_1|11\rangle, \quad (1)$$

having Schmidt coefficients l_0, l_1 (with $0 \leq l_1 \leq l_0 \leq 1$ and $l_0^2 + l_1^2 = 1$) and Schmidt basis $\{|0\rangle_i, |1\rangle_i\}$ for the i -th particle ($i = 1, 2$). Here the given amount of entanglement is $E = -l_0^2 \log_2 l_0^2 - l_1^2 \log_2 l_1^2$. Let $\mathcal{S}_{(l_0, l_1)} = \{(U \otimes V)|\psi_0\rangle : U, V \in SU(2)\}$ be the collection of all the two-qubit pure states, each having entanglement E .

2.1 Constant rotation of equally entangled two-qubit pure states by one-sided local unitary operators

Let us now look for an one-sided local unitary operator $W_1 \otimes I_2 \in SU(2) \otimes SU(2)$ (or $I_2 \otimes W_2 \in SU(2) \otimes SU(2)$) which can rotate $|\psi_0\rangle$ as well as some or all the other elements of $\mathcal{S}_{(l_0, l_1)}$ through one and the same given amount $re^{i\theta}$ (with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$). Taking $W_1 = r_0^{W_1} I_2 + i\vec{r}^{\overline{W_1}} \cdot \vec{\sigma}$ (with $(r_0^{W_1}, \vec{r}^{\overline{W_1}}) \in \mathbb{R}^2$ and $(r_0^{W_1})^2 + |\vec{r}^{\overline{W_1}}|^2 = 1$), we get from the condition $\langle\psi_0|W_1 \otimes I_2|\psi_0\rangle = re^{i\theta}$ that $r_0^{W_1} = r \cos \theta$ and $r_z^{W_1}(l_0^2 - l_1^2) = r \sin \theta$. Thus, we are looking for the maximal subset $\mathcal{S}_{W_1 \otimes I_2}^{(l_0, l_1)}$ of states $(U \otimes V)|\psi_0\rangle$ from $\mathcal{S}_{(l_0, l_1)}$ for which $\langle\psi_0|U^\dagger W_1 U \otimes I_2|\psi_0\rangle = re^{i\theta} \equiv r_0^{W_1} + i\vec{r}_z^{W_1}(l_0^2 - l_1^2)$. Now $U^\dagger W_1 U = r_0^{W_1} I_2 + i(R_{U^\dagger} \vec{r}^{\overline{W_1}}) \cdot \vec{\sigma}$, where, for any $U \in SU(2)$, R_U is the 3×3 real orthogonal rotation matrix corresponding to U . In other words, for any element $\vec{a} \in \mathbb{R}^3$, we have

$$R_U \vec{a} \equiv 2(\vec{r}^{\overline{U}} \cdot \vec{a}) \vec{r}^{\overline{U}} + (1 - 2|\vec{r}^{\overline{U}}|^2) \vec{a} - 2r_0^U (\vec{r}^{\overline{U}} \times \vec{a}). \quad (2)$$

So we have $\langle\psi_0|U^\dagger W_1 U \otimes I_2|\psi_0\rangle = r_0^{W_1} + i(R_{U^\dagger} \vec{r}^{\overline{W_1}})_z (l_0^2 - l_1^2)$. Thus we are looking for the *maximal* set of $(U \otimes V)$'s from $SU(2) \otimes SU(2)$ so that for each such $U \otimes V$, we have

$$(l_0^2 - l_1^2)[(R_{U^\dagger} \vec{r}^{\overline{W_1}})_z - r_z^{W_1}] = 0. \quad (3)$$

If $|\psi_0\rangle$ is a maximally entangled state, condition (3) is thus satisfied for *all* $U, V \in SU(2)$, *i.e.*, for all two-qubit maximally entangled states. If $|\psi_0\rangle$ is not a maximally entangled

state, then the desired maximal set will consist of only those states $(U \otimes V)|\psi_0\rangle$ for which V can be an *arbitrary* element of $SU(2)$ but U will be such that Bloch vectors $R_{U^\dagger}(\vec{r}^{W_1}/|\vec{r}^{W_1}|)$ span a small circle of the Bloch sphere whose plane is perpendicular to \hat{z} and is at a distance $r_z^{W_1}/|\vec{r}^{W_1}|$ from the centre of the Bloch sphere. So R_{U^\dagger} corresponds to the unitary operator $\exp[-i(\theta/2)\sigma_z] = \cos(\theta/2)I_2 - i\sin(\theta/2)\sigma_z$, *i.e.*, rotation about the z -axis through some angle θ .

Rotation by one-sided local unitary operators of the form $I_2 \otimes W_2$ will provide the similar result.

Note that by a single unitary operator $U \in SU(2)$, one can rotate, through the same angle, only those single-qubit pure states whose Bloch vectors lie on a particular (depending upon U) circle of the Bloch sphere.

2.2 Constant rotation of two-qubit pure states by one-sided local unitary operators

Another way to rotate two-qubit pure states by one-sided local unitaries of the form $(W_1 \otimes I_2) \in SU(2) \otimes SU(2)$, through the same amount, is to first consider the set $\mathcal{S}(W_1)$ of all single-qubit pure states $|\psi(\theta, \phi)\rangle \equiv \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \equiv U|0\rangle \equiv (\cos(\theta/2)I_2 + i\sin\phi\sin(\theta/2)\sigma_x - i\cos\phi\sin(\theta/2)\sigma_y)|0\rangle$, each of which can be rotated through the same amount $\langle 0|W_1|0\rangle = r_0^{W_1} + r_z^{W_1}$ by the unitary matrix W_1 . Note that

$$\mathcal{S}(W_1) = \left\{ |\psi(\theta, \phi)\rangle : \hat{v}(\theta, \phi) \cdot \vec{r}^{W_1} = r_z^{W_1} \right\},$$

where $\hat{v}(\theta, \phi) \equiv (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ is the Bloch vector of the state $|\psi(\theta, \phi)\rangle$. Thus $\mathcal{S}(W_1)$ is a circular section (generally, a small circle) of the Bloch sphere, perpendicular to the vector \vec{r}^{W_1} , while the projection of each Bloch vector of that circle along \vec{r}^{W_1} is same as $r_z^{W_1}$. It is then easy to show that all the members of the following set of two-qubit pure states

$$\mathcal{S}_{W_1} \equiv \left\{ |\chi\rangle = \sqrt{\lambda}|\psi(\theta_1, \phi_1)\rangle \otimes |e\rangle + \sqrt{1-\lambda}|\psi(\theta_2, \phi_2)\rangle \otimes |e^\perp\rangle : |\psi(\theta_1, \phi_1)\rangle, |\psi(\theta_2, \phi_2)\rangle \in \right.$$

$$\left. \mathcal{S}(W_1), \lambda \in [0, 1], \{|e\rangle, |e^\perp\rangle\} \text{ is an ONB of } \mathcal{C}^2 \right\}$$

can be rotated through the same amount $r_0^{W_1} + r_z^{W_1}$ by the one-sided local unitary $(W_1 \otimes I_2)$. Note that the single-qubit reduced density matrix $\rho_1^\chi \equiv \text{Tr}_2[|\chi\rangle\langle\chi|] = (1/2)[I_2 + \{\lambda\hat{v}(\theta_1, \phi_1) + (1-\lambda)\hat{v}(\theta_2, \phi_2)\} \cdot \vec{\sigma}]$ of $|\chi\rangle$ has eigen values $(1 \pm |\vec{R}|)/2$, where $|\vec{R}|^2 \equiv |\lambda\hat{v}(\theta_1, \phi_1) + (1-\lambda)\hat{v}(\theta_2, \phi_2)|^2 = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)(\hat{v}(\theta_1, \phi_1) \cdot \hat{v}(\theta_2, \phi_2)) = 1 - 2\lambda(1-\lambda)(1 - \hat{v}(\theta_1, \phi_1) \cdot \hat{v}(\theta_2, \phi_2))$. *So, not all the states $|\chi\rangle \in \mathcal{S}_{W_1}$ are equally entangled.* In the special case when $\hat{v}(\theta_1, \phi_1) \cdot \hat{v}(\theta_2, \phi_2) = -1$ (which happens only when $\mathcal{S}(W_1)$ is a great circle, *i.e.*, when W_1 is a NOT operation), for arbitrary but fixed $\lambda \in [0, 1]$, each member of the set of all the *equally entangled* two-qubit pure states $\sqrt{\lambda}|\psi(\theta, \phi)\rangle \otimes |e\rangle + \sqrt{1-\lambda}|\psi(\pi-\theta, \pi+\phi)\rangle \otimes |e^\perp\rangle$ can be rotated to its orthogonal state by the one-sided local

unitary $(W_1 \otimes I_2)$, where $\{|e\rangle, |e^\perp\rangle\}$ is any ONB of \mathcal{C}^2 . For any given Schmidt coefficients $l_0 = \sqrt{1 - \lambda(1 - \lambda)(1 - \hat{v}(\theta_1, \phi_1) \cdot \hat{v}(\theta_2, \phi_2))}$ and $l_1 = \sqrt{\lambda(1 - \lambda)(1 - \hat{v}(\theta_1, \phi_1) \cdot \hat{v}(\theta_2, \phi_2))}$, all the members of the subset $(\mathcal{S}'_{W_1}^{(l_0, l_1)})$ (say) of \mathcal{S}_{W_1} of all the two-qubit pure states $|\chi\rangle$, each having one and the same pair of Schmidt coefficients (l_0, l_1) , can be rotated through the same amount $\langle 0|W_1|0\rangle$ by the local unitary operator $W_1 \otimes I_2$, where as, all the members of the set $\mathcal{S}_{W_1 \otimes I_2}^D = \{|UDV^T\rangle : V \in SU(2) \text{ and } T_{1z} = 0\}$, with $D = \text{diag}(l_0, l_1)$ and

$$\vec{T}_1 = \left| \vec{r}^{\vec{U}} \right|^2 \vec{r}^{\vec{W}_1} - \left(\vec{r}^{\vec{U}} \cdot \vec{r}^{\vec{W}_1} \right) \vec{r}^{\vec{U}} - r_0^U \left(\vec{r}^{\vec{U}} \times \vec{r}^{\vec{W}_1} \right),$$

can be rotated⁷ by $W_1 \otimes I_2$ through the same amount $\langle \psi_0 | (W_1 \otimes I_2) | \psi_0 \rangle$ (where $|\psi_0\rangle = l_0|00\rangle + l_1|11\rangle$). By mere parameter counting, one can see that $\mathcal{S}'_{W_1}^{(l_0, l_1)}$ can be specified by *at most four independent real parameters*, while $\mathcal{S}_{W_1 \otimes I_2}^D$ can be specified by *at most five independent real parameters*. Hence, from this perspective, the size of $\mathcal{S}'_{W_1}^{(l_0, l_1)}$ seems to be smaller than that of $\mathcal{S}_{W_1 \otimes I_2}^D$.

2.3 Constant rotation of equally entangled two-qubit pure states by two-sided local unitary operators

Now we come to the issue of enlarging the maximal set $\mathcal{S}_{W_1 \otimes I_2}^{(l_0, l_1)}$ (given by equation (3)) by considering rotations using both-sided local unitary operators $W_1 \otimes W_2 \in SU(2) \otimes SU(2)$. Both-sided local unitary operators, in general, give rise to maximal sets whose sizes are different from that for one-sided local unitary operations.

In fact, here we look for the maximal subset $\mathcal{S}_{W_1 \otimes W_2}^{(l_0, l_1)}$ of the elements $(U \otimes V)|\psi_0\rangle$ of $\mathcal{S}_{(l_0, l_1)}$ for which $\langle \psi_0 | U^\dagger W_1 U \otimes V^\dagger W_2 V | \psi_0 \rangle = \langle \psi_0 | W_1 \otimes I_2 | \psi_0 \rangle$, i.e.,

$$\begin{aligned} r_0^{W_1} (R_{U^\dagger} \vec{r}^{\vec{W}_1})_z + r_0^{W_2} (R_{V^\dagger} \vec{r}^{\vec{W}_2})_z + 2l_0 l_1 \{ (R_{U^\dagger} \vec{r}^{\vec{W}_1})_x (R_{V^\dagger} \vec{r}^{\vec{W}_2})_x - (R_{U^\dagger} \vec{r}^{\vec{W}_1})_y (R_{V^\dagger} \vec{r}^{\vec{W}_2})_y \} + (R_{U^\dagger} \vec{r}^{\vec{W}_1})_z (R_{V^\dagger} \vec{r}^{\vec{W}_2})_z \\ = r_0^{W_1} r_z^{W_2} + r_0^{W_2} r_z^{W_1} + 2l_0 l_1 \{ r_x^{W_1} r_x^{W_2} - r_y^{W_1} r_y^{W_2} \} + r_z^{W_1} r_z^{W_2}. \end{aligned} \quad (4)$$

To get an idea about the maximal set, let us consider the example where $W_1 = W_2 = (i/\sqrt{2})(\sigma_x + \sigma_y)$. So the above-mentioned condition (4) now becomes $2l_0 l_1 \{ (R_{U^\dagger} \vec{r}^{\vec{W}_1})_x (R_{V^\dagger} \vec{r}^{\vec{W}_2})_x - (R_{U^\dagger} \vec{r}^{\vec{W}_1})_y (R_{V^\dagger} \vec{r}^{\vec{W}_2})_y \} + (R_{U^\dagger} \vec{r}^{\vec{W}_1})_z (R_{V^\dagger} \vec{r}^{\vec{W}_2})_z = 0$, which, in turn, implies that the two vectors $\vec{T}_1(U; l_0, l_1) \equiv 2l_0 l_1 (R_{U^\dagger} \vec{r}^{\vec{W}_1})_x \hat{x} - 2l_0 l_1 (R_{U^\dagger} \vec{r}^{\vec{W}_1})_y \hat{y} + (R_{U^\dagger} \vec{r}^{\vec{W}_1})_z \hat{z} \equiv 2l_0 l_1 (R_{U^\dagger} \vec{r}^{\vec{W}_1})_x \hat{x} + 2l_0 l_1 (R_{U^\dagger} \vec{r}^{\vec{W}_1})_y \hat{y} + (R_{U^\dagger} \vec{r}^{\vec{W}_1})_z \hat{z}$ and $\vec{T}_2(V) \equiv R_{V^\dagger} \vec{r}^{\vec{W}_2}$ are orthogonal. In other words, for arbitrarily given $V \in SU(2)$, $U \in SU(2)$ should be such that the Bloch vector $\vec{T}_1(U; l_0, l_1)/|\vec{T}_1(U; l_0, l_1)|$ lies on the great circle $(\mathcal{G}(\vec{T}_2(V)))$, say, orthogonal to the Bloch vector $\vec{T}_2(V)/|\vec{T}_2(V)|$. So, for every $V \in SU(2)$, $R_{U^\dagger} \vec{r}^{\vec{W}_1}$ will an arbitrary Bloch vector lying on the great circle perpendicular to the Bloch vector $R_{V^\dagger} \vec{r}^{\vec{W}_2}$. In other words, for every $V \in SU(2)$, R_{U^\dagger} implies rotation (through some angle) about the vector $R_{V^\dagger} \vec{r}^{\vec{W}_2}$. Thus we see that the size of the required maximal set $\mathcal{S}_{W_1 \otimes W_2}^{(l_0, l_1)}$ is *same* as in the case of one-sided local unitary operations.

⁷For definition of $|UDV^T\rangle$, please look at the last but one paragraph of the conclusion section.

3 Constant rotation of equally entangled two-qubit pure states by one-sided unital trace-preserving CP maps

Considering the action of the local quantum operations at the density matrix level, we may look for the maximal set $\mathcal{T}_{W_1 \otimes W_2}^{(l_0, l_1)} \equiv \{(U \otimes V)|\psi_0\rangle \in \mathcal{S}_{(l_0, l_1)} : \langle\psi_0|(U^\dagger \otimes V^\dagger)\{(W_1 \otimes W_2)((U \otimes V)|\psi_0\rangle\langle\psi_0|(U^\dagger \otimes V^\dagger))(W_1^\dagger \otimes W_2^\dagger)\}(U \otimes V)|\psi_0\rangle = \langle\psi_0|\{(W_1 \otimes W_2)|\psi_0\rangle\langle\psi_0|(W_1^\dagger \otimes W_2^\dagger)\}|\psi_0\rangle\} = \{(U \otimes V)|\psi_0\rangle \in \mathcal{S}_{(l_0, l_1)} : |\langle\psi_0|U^\dagger W_1 U \otimes V^\dagger W_2 V|\psi_0\rangle|^2 = |\langle\psi_0|W_1 \otimes W_2|\psi_0\rangle|^2\}$, instead of the maximal set $\mathcal{S}_{W_1 \otimes W_2}^{(l_0, l_1)}$. This immediately shows that $\mathcal{S}_{W_1 \otimes W_2}^{(l_0, l_1)} \subset \mathcal{T}_{W_1 \otimes W_2}^{(l_0, l_1)}$. Thus, for example, $\mathcal{T}_{W_1 \otimes I_2}^{(l_0, l_1)}$ will consist of all the states $(U \otimes V)|\psi_0\rangle \in \mathcal{S}_{(l_0, l_1)}$ where V is an arbitrary element of $SU(2)$ while $U \in S(2)$ is such that R_{U^\dagger} can be either a rotation about the positive z axis or a rotation about the negative z axis.

We now raise the issue of further enlargement of the maximal set $\mathcal{T}_{W_1 \otimes W_2}^{(l_0, l_1)}$ by using LOCC, not just local unitary operators. Here, for simplicity, we consider only the question of rotating, through the same amount, all or some of the elements of $\mathcal{S}_{(l_0, l_1)}$ by using an one-sided local unital CP map $T_1 \otimes I_2$, where T_1 is a unital, trace-preserving CP map on $\mathcal{B}(\mathcal{C}^2)$. Any unital trace-preserving CP map $T_1 : \mathcal{B}(\mathcal{C}^2) \rightarrow \mathcal{B}(\mathcal{C}^2)$ can be expressed as $T_1(\rho) = \sum_{j=1}^4 \lambda_j W'_j \rho W'_j{}^\dagger$ where $\rho \in \mathcal{B}(\mathcal{C}^2)$, $W'_j \in SU(2)$, $0 \leq \lambda_j \leq 1$ (for $j = 1, 2, 3, 4$), and $\sum_{j=1}^4 \lambda_j = 1$ [11]. Thus we are looking for the maximal subset $\mathcal{T}_{T_1 \otimes I_2}^{(l_0, l_1)}$ of states $(U \otimes V)|\psi_0\rangle$ (including the state $|\psi_0\rangle$) from $\mathcal{S}_{(l_0, l_1)}$ for which

$$\begin{aligned} & \sum_{i=1}^4 \lambda_i \langle\psi_0|(U^\dagger W'_i U \otimes I_2)|\psi_0\rangle \langle\psi_0|(U^\dagger W'_i U \otimes I_2)^\dagger|\psi_0\rangle \\ &= \sum_{i=1}^4 \lambda_i \langle\psi_0|(W'_i \otimes I_2)|\psi_0\rangle \langle\psi_0|(W'_i \otimes I_2)^\dagger|\psi_0\rangle. \end{aligned}$$

We are thus looking for all those $U, V \in SU(2)$ for which

$$\begin{aligned} & l_0^2 \sum_{j=1}^4 \lambda_j |\langle\psi_0|\{((U^\dagger W'_j U)|0\rangle) \otimes |0\rangle\}|^2 + l_1^2 \sum_{j=1}^4 \lambda_j |\langle\psi_0|\{((U^\dagger W'_j U)|1\rangle) \otimes |1\rangle\}|^2 + \\ & 2l_0 l_1 \sum_{j=1}^4 \lambda_j \text{Re}([\langle\psi_0|\{((U^\dagger W'_j U)|0\rangle) \otimes |0\rangle\}] \times [\langle\psi_0|\{((U^\dagger W'_j U)|1\rangle) \otimes |1\rangle\}]^*) = l_0^2 \sum_{j=1}^4 \lambda_j |\langle\psi_0|\{(W'_j|0\rangle) \otimes |0\rangle\}|^2 + \\ & l_1^2 \sum_{j=1}^4 \lambda_j |\langle\psi_0|\{(W'_j|1\rangle) \otimes |1\rangle\}|^2 + 2l_0 l_1 \sum_{j=1}^4 \lambda_j \text{Re}([\langle\psi_0|\{(W'_j|0\rangle) \otimes |0\rangle\}] \times [\langle\psi_0|\{(W'_j|1\rangle) \otimes |1\rangle\}]^*), \end{aligned}$$

which is nothing but the following condition:

$$(l_0^2 - l_1^2)^2 \sum_{j=1}^4 \lambda_j \left[\left\{ \left(R_{U^\dagger} \overrightarrow{r^{W'_j}} \right)_z \right\}^2 - \left(r_z^{W'_j} \right)^2 \right] = 0. \quad (5)$$

For a maximally entangled state $|\psi_0\rangle$, this condition is *automatically satisfied for all* $U, V \in SU(2)$. When $|\psi_0\rangle$ is a non-maximally entangled state, the above-mentioned condition becomes

$$\sum_{j=1}^4 \lambda_j \left[\left\{ \left(R_{U^\dagger} r_z^{\overrightarrow{W'_j}} \right)_z \right\}^2 - \left(r_z^{W'_j} \right)^2 \right] = 0. \quad (6)$$

If R_{U^\dagger} represents a rotation about the z -axis of the Bloch sphere, then $(R_{U^\dagger} r_z^{\overrightarrow{W'_j}})_z = r_z^{W'_j}$ for $j = 1, 2, 3, 4$, and so, condition (6) will be satisfied in this case. We have seen this solution earlier in the case of constant rotation by one-sided local unitary transformation $W_1 \otimes I_2$. But equation (6) will have more solution(s). This implies that the size of $\mathcal{T}_{W_1 \otimes I_2}^{(l_0, l_1)}$ will be *smaller*, in general, than the size of set $\mathcal{T}_{T_1 \otimes I_2}^{(l_0, l_1)}$.

In the case of the bit-flip channel $T_1(\rho) \equiv p\rho + (1-p)\sigma_x\rho\sigma_x$ (with $0 \leq p < 1$), we have $\lambda_1 = p$, $\lambda_2 = (1-p)$, $\lambda_3 = \lambda_4 = 0$ and $W'_1 = I_2$, $W'_2 = i\sigma_x$. Then the above-mentioned condition becomes $(1-p)\{(R_{U^\dagger}\hat{x})_z\}^2 = 0$, *i.e.*, $(R_{U^\dagger}\hat{x})_z = 0$. So, as in the case of equal rotations by one-sided local unitary operators, here R_{U^\dagger} will correspond to rotations about the z -axis of the Bloch sphere.

On the other hand, for the depolarizing channel $T_1(\rho) \equiv p\rho + ((1-p)/3)[\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z]$ (with $0 \leq p < 1$ and $1-p$ is called the ‘depolarization coefficient’), we have $\lambda_1 = p$, $\lambda_2 = \lambda_3 = \lambda_4 = (1-p)/3$ and $W'_1 = I_2$, $W'_2 = i\sigma_x$, $W'_3 = i\sigma_y$, $W'_4 = i\sigma_z$. Then the above-mentioned condition becomes $1 = \{(R_{U^\dagger}\hat{x})_z\}^2 + \{(R_{U^\dagger}\hat{y})_z\}^2 + \{(R_{U^\dagger}\hat{z})_z\}^2$. This condition is *always satisfied whatever be the unitary operator* $U \in SU(2)$ (use equation (2)). Thus we see that if we want to rotate, through one and the same amount $r \in [0, 1]$ (with $r \geq (l_0^2 - l_1^2)^2/3$), the state $|\psi_0\rangle$ as well as all the other members $(U \otimes V)|\psi_0\rangle$ of $\mathcal{S}_{(l_0, l_1)}$ by some one-sided local unital trace-preserving CP map $(T_1 \otimes I) : \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2) \rightarrow \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{C}^2)$ (in the sense that $\langle\psi_0|(U^\dagger \otimes V^\dagger)[(T_1 \otimes I)((U \otimes V)|\psi_0\rangle\langle\psi_0|(U^\dagger \otimes V^\dagger))](U \otimes V)|\psi_0\rangle = \langle\psi_0|[(T_1 \otimes I)(|\psi_0\rangle\langle\psi_0|)]|\psi_0\rangle \equiv r$), we can do so by taking T_1 as the depolarizing channel with $p = \{3r - (l_0^2 - l_1^2)^2\}/\{3 - (l_0^2 - l_1^2)^2\}$. So, when $|\psi_0\rangle$ is not a maximally entangled state, the one-sided local depolarizing channel $(T_1 \otimes I)$ *can not* act as a NOT operation on the elements of set of all the equally entangled states $(U \otimes V)|\psi_0\rangle$. Once again, *every* maximally entangled state $|\psi\rangle$ of two qubits can be transformed into a two-qubit state ρ , having support in the orthogonal subspace of $|\psi\rangle$, by the one-sided local depolarizing map $(T_1 \otimes I)$.

Note that by the action of the depolarizing map T_1 on all the single-qubit pure states $U|0\rangle$ (where $U \in SU(2)$), every such state can be state can be ‘rotated’, through the same amount $\langle 0|T_1(|0\rangle\langle 0|)|0\rangle = (1+2p)/3$. But this map can not act as a quantum NOT operation as here $1/3 \leq (1+2p)/3 < 1$.

4 Local rotation of three-qubit maximally entangled states

Let us consider the three-qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (7)$$

This is a maximally entangled state of three qubits. It is to be noted that an n -qubit pure state is said to be a maximally entangled state if and only if it has maximal available entanglement in every bipartite cut. According to this definition, every maximally entangled state $|\psi_{max}\rangle$ of three qubits is connected to $|GHZ\rangle$ via local unitary of the form $U_A(\alpha_1, \gamma_1, \delta_1) \otimes U_B(\alpha_2, \gamma_2, \delta_2) \otimes U_C(\alpha_3, \gamma_3, \delta_3)$ and vice-versa, where (with respect to the basis $\{|0\rangle, |1\rangle\}$)

$$U_A(\alpha_1, \gamma_1, \delta_1) = \begin{pmatrix} e^{i\delta_1} \cos\alpha_1 & e^{i\gamma_1} \sin\alpha_1 \\ -e^{-i\gamma_1} \sin\alpha_1 & e^{-i\delta_1} \cos\alpha_1 \end{pmatrix}, \quad (8)$$

with $\alpha_1, \gamma_1, \delta_1 \in [0, 2\pi]$. Similar form for the others unitary operators $U_B(\alpha_2, \gamma_2, \delta_2)$ and $U_C(\alpha_3, \gamma_3, \delta_3)$. One can also show that

$$\begin{aligned} |\psi_{max}\rangle &= (U(\alpha_1, \gamma_1, \delta_1) \otimes U(\alpha_2, \gamma_2, \delta_2) \otimes U(\alpha_3, \gamma_3, \delta_3)) |GHZ\rangle \\ &= (I \otimes V_{23}(\alpha_1, \gamma_1, \delta_1; \alpha_2, \gamma_2, \delta_2; \alpha_3, \gamma_3, \delta_3)) |GHZ\rangle, \end{aligned} \quad (9)$$

where $V_{23}(\alpha_1, \gamma_1, \delta_1; \alpha_2, \gamma_2, \delta_2; \alpha_3, \gamma_3, \delta_3)$ is a two-qubit unitary matrix, acting on $|00\rangle$ and $|11\rangle$ in the following way

$$\begin{aligned} V_{23}|00\rangle &= e^{i\delta_1} \cos\alpha_1 |\eta(\alpha_2, \gamma_2, \delta_2) \eta(\alpha_3, \gamma_3, \delta_3)\rangle + e^{i\gamma_1} \sin\alpha_1 \overline{|\eta(\alpha_2, \gamma_2, \delta_2) \eta(\alpha_3, \gamma_3, \delta_3)\rangle}, \\ V_{23}|11\rangle &= -e^{-i\gamma_1} \sin\alpha_1 |\eta(\alpha_2, \gamma_2, \delta_2) \eta(\alpha_3, \gamma_3, \delta_3)\rangle + e^{-i\delta_1} \cos\alpha_1 \overline{|\eta(\alpha_2, \gamma_2, \delta_2) \eta(\alpha_3, \gamma_3, \delta_3)\rangle}, \end{aligned} \quad (10)$$

with

$$\begin{aligned} |\eta(\alpha_2, \gamma_2, \delta_2)\rangle &= e^{i\delta_2} \cos\alpha_2 |0\rangle - e^{-i\gamma_2} \sin\alpha_2 |1\rangle, \\ \overline{|\eta(\alpha_2, \gamma_2, \delta_2)\rangle} &= e^{i\gamma_2} \sin\alpha_2 |0\rangle + e^{-i\delta_2} \cos\alpha_2 |1\rangle. \end{aligned} \quad (11)$$

Similar expression holds for the other states $|\eta(\alpha_3, \gamma_3, \delta_3)\rangle$.

Given $\theta, \phi \in [0, 2\pi]$, let us now try to find out a three-qubit quantum operation \mathcal{A} such that

$$\langle\psi_{max}|\mathcal{A}|\psi_{max}\rangle = e^{i\phi} \cos\theta \quad (12)$$

for *largest* number of three-qubit maximally entangled states $|\psi_{max}\rangle$. One can *always* do so by choosing $\alpha', \gamma', \delta'$ in $[0, 2\pi]$ properly such that $\mathcal{A} = e^{i\phi} U(\alpha', \gamma', \delta') \otimes I \otimes I$, where $U(\alpha', \gamma', \delta')$ is given in equation (8) and I is the single-qubit identity operator. In fact, we have

$$\langle GHZ|(e^{i\phi} U(\alpha', \gamma', \delta') \otimes I \otimes I)|GHZ\rangle = e^{i\phi} \cos\alpha' \cos\delta' \equiv e^{i\phi} \cos\theta. \quad (13)$$

And (using equation (9))

$$\langle\psi_{max}|(e^{i\phi} U(\alpha', \gamma', \delta') \otimes I \otimes I)|\psi_{max}\rangle =$$

$$\begin{aligned}
\langle GHZ | \left(e^{i\phi} U(\alpha', \gamma', \delta') \otimes (V_{23}(\alpha_1, \gamma_1, \delta_1; \alpha_2, \gamma_2, \delta_2; \alpha_3, \gamma_3, \delta_3))^{\dagger} V_{23}(\alpha_1, \gamma_1, \delta_1; \alpha_2, \gamma_2, \delta_2; \alpha_3, \gamma_3, \delta_3) \right) | GHZ \rangle \\
= \langle GHZ | (e^{i\phi} U(\alpha', \gamma', \delta') \otimes I \otimes I) | GHZ \rangle \\
= e^{i\phi} \cos \alpha' \cos \delta'.
\end{aligned} \tag{14}$$

Thus we see that given $\theta, \phi \in [0, 2\pi]$, one can always find out a single-qubit unitary matrix $U(\alpha', \gamma', \delta')$ (given by equation (8)) such that $\langle \psi_{max} | (e^{i\phi} U(\alpha', \gamma', \delta') \otimes I \otimes I) | \psi_{max} \rangle = e^{i\phi} \cos \theta$ for *all* three-qubit maximally entangled states $|\psi_{max}\rangle$.

5 Conclusion

It is shown that all two-qubit pure maximally entangled states can be rotated by a constant amount via local unitary operators of the forms $W_1 \otimes I_2$ or $I_2 \otimes W_2$. But there exists no local unitary operator which can rotate all non-maximally entangled two-qubit pure states for a fixed degree of entanglement. We have found the optimal sets of non-maximally entangled states for a fixed degree of entanglement, each of which can be rotated through a constant amount by local unitary operations of three different forms $W_1 \otimes W_2$, $W_1 \otimes I_2$ and $I_2 \otimes W_2$. In particular, when this amount is zero, we get the local quantum NOT operation. Although the sizes of the optimal sets are, in general, different for one-sided and two-sided local unitary operations, the size of the optimal set for two-sided local quantum NOT operation is same as that for any one-sided local unitary operation. Surprisingly we found that the depolarizing map on one of the two qubits, all equally entangled two-qubit pure states can be rotated through a constant amount, whose value depends on the value of the depolarization coefficient. Our result for two-qubits is extended for the case of three-qubit maximally entangled state. Note that Novotný et al. [10] have considered, most of times, covariant trace-preserving two-qubit CP maps which would serve as quantum NOT operations for the set of some or all two-qubit equally entangled pure states. This covariance allows only the set of all maximally entangled states to be transformed into their respective orthogonal states. For non-maximally entangled states (all of which are equally entangled), this covariance allows only for *approximate* NOT operations. *None* of the operations, we have considered in this work, satisfies this covariance, although each of them is a unital trace-preserving CP map.

In this paper, we have noticed that so far as rotation of two-qubit pure states, through the same amount by one-sided local quantum operations, are concerned, the size of the maximal set of two-qubit pure states will be *same* as that of the single-qubit pure states, provided we restrict our attention to only equally entangled two-qubit pure states.

So far, we have discussed only about LOCC (more specifically, local operations only). In order to have an idea about the difference in the sizes of the maximal sets $\mathcal{S}_{W_1 \otimes W_2}^{(l_0, l_1)}$ and $\mathcal{S}_W^{(l_0, l_1)}$, respectively for local rotator $W_1 \otimes W_2$ and non-local rotator W , let us consider the swap operator $W_{swap} = (1/2)[I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z] \in U(4)$, so that $W \equiv e^{i\pi/4} W_{swap} \in SU(4)$ is a non-local special unitary operator. Here we shall use the fact that $W_{swap} = 2(|\phi^+\rangle_{AB} \langle \phi^+|)^{TB}$, where $|\phi^+\rangle_{AB} = (1/\sqrt{2})(|00\rangle_{AB} + |11\rangle_{AB})$ and T_B denotes partial transposition with respect to the qubit B . Now $\langle \psi_0 | (U^\dagger \otimes V^\dagger) W (U \otimes$

$V)|\psi_0\rangle = \text{Tr}[(U^\dagger \otimes V^\dagger)W(U \otimes V)|\psi_0\rangle\langle\psi_0|] = \text{Tr}[(U^\dagger \otimes V^*)W^{T_B}(U \otimes V^T)(|\psi_0\rangle\langle\psi_0|)^{T_B}] = 2e^{i\pi/4}\text{Tr}[(U^\dagger \otimes V^*)|\phi^+\rangle\langle\phi^+|(U \otimes V^T)(|\psi_0\rangle\langle\psi_0|)^{T_B}] = 2e^{i\pi/4}\langle\phi^+|(U \otimes V^T)\{(|\psi_0\rangle_{(l_0,l_1)}\langle\psi_0|)^{T_B}\}(U^\dagger \otimes V^*)|\phi^+\rangle = 2e^{i\pi/4}[l_0^2|\langle\phi^+|(U \otimes V^T)|00\rangle|^2 + l_1^2|\langle\phi^+|(U \otimes V^T)|11\rangle|^2 + l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^+\rangle|^2 - l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^-\rangle|^2],$ where $|\psi^\pm\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$. So, we are looking for all possible $U, V \in SU(2)$ for which $2e^{i\pi/4}[l_0^2|\langle\phi^+|(U \otimes V^T)|00\rangle|^2 + l_1^2|\langle\phi^+|(U \otimes V^T)|11\rangle|^2 + l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^+\rangle|^2 - l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^-\rangle|^2] = 2e^{i\pi/4}[l_0^2|\langle\phi^+|00\rangle|^2 + l_1^2|\langle\phi^+|11\rangle|^2 + l_0l_1|\langle\phi^+|\psi^+\rangle|^2 - l_0l_1|\langle\phi^+|\psi^-\rangle|^2]$, i.e., $l_0^2|\langle\phi^+|(U \otimes V^T)|00\rangle|^2 + l_1^2|\langle\phi^+|(U \otimes V^T)|11\rangle|^2 + l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^+\rangle|^2 - l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^-\rangle|^2 = 1/2$. Now the maximum value, the quantity $l_0^2|\langle\phi^+|(U \otimes V^T)|00\rangle|^2 + l_1^2|\langle\phi^+|(U \otimes V^T)|11\rangle|^2 + l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^+\rangle|^2 - l_0l_1|\langle\phi^+|(U \otimes V^T)|\psi^-\rangle|^2$ can attain, is $1/2$, which occurs if $|\langle\phi^+|(U \otimes V^T)|00\rangle|^2 = |\langle\phi^+|(U \otimes V^T)|11\rangle|^2 = 1/2$ and $|\langle\phi^+|(U \otimes V^T)|\psi^+\rangle|^2 = |\langle\phi^+|(U \otimes V^T)|\psi^-\rangle|^2 = 0$. In this case V (conversely, U) can be chosen *arbitrarily* from $SU(2)$ but U has to be chosen from $SU(2)$ in such a way that $r_x^U = r_x^V$ and $r_y^U = r_y^V$. Note that this is true *irrespective* of whether $|\psi_0\rangle$ is maximally entangled or not. Thus we see that the swap operator can not rotate, with equal amount, all the maximally entangled states. Nevertheless, in the case of a non-maximally entangled state $|\psi_0\rangle$, the maximal set $\mathcal{S}_{e^{i\pi/4}W_{\text{swap}}}^{(l_0,l_1)}$ is *much* larger than the corresponding maximal set $\mathcal{S}_{W_1 \otimes I_2}^{(l_0,l_1)}$ (or $\mathcal{S}_{I_2 \otimes W_2}^{(l_0,l_1)}$) for one-sided local unitary operators.

As in the case of $2 \otimes 2$ systems, all maximally entangled states of a $d \otimes d$ system can be rotated, through arbitrary but same amount, by one-sided local unitary operators. The question of rotating non-maximally entangled states of $d \otimes d$ is more subtle as we don't have a Bloch sphere representation for d dimensional quantum systems when $d > 2$. In fact, in the standard product basis $\{|ij\rangle : i, j = 0, 1, \dots, d-1\}$ of $\mathcal{C}^d \otimes \mathcal{C}^d$, every normalized pure state $|\psi\rangle = \sum_{i,j=0}^{d-1} c_{ij}|ij\rangle$ can be represented by the $d \times d$ matrix $C \equiv (c_{ij})_{i,j=0}^{d-1}$ (where $\text{Tr}[CC^\dagger] = 1$), or, equivalently, by the symbol $|C\rangle\rangle$. Given the non-negative diagonal matrix $D = \text{diag}(D_0, D_1, \dots, D_{d-1})$ (with $0 \leq D_{d-1} \leq D_{d-2} \leq \dots \leq D_0 \leq 1$ and $\sum_{i=0}^{d-1} D_i^2 = 1$), we would like to find out the maximal set of all the equally entangled states $|\psi\rangle \equiv (U \otimes V)|D\rangle\rangle = |UDV^T\rangle\rangle \in \mathcal{C}^d \otimes \mathcal{C}^d$ (including the state $|D\rangle\rangle$) for which $\langle\langle UDV^T|(W_1 \otimes W_2)|UDV^T\rangle\rangle = \langle\langle D|(W_1 \otimes W_2)|D\rangle\rangle$, i.e., $\text{Tr}[W_1(UDV^T)W_2^T(UDV^T)^\dagger] = \text{Tr}[W_1DW_2^TD]$, where W_1, W_2 are fixed elements of $SU(d)$ while U, V can be some or all the elements of $SU(d)$. When $W_2 = I_d$, this condition becomes $\text{Tr}[W_1UD^2U^\dagger] = \text{Tr}[W_1D^2]$, which will be satisfied for all those $U \in SU(d)$, each of which commutes with W_1 (but these are not the only solutions). When $|D\rangle\rangle$ is a maximally entangled state, $D = (1/\sqrt{d})I_d$, and so the last condition will be automatically satisfied for this D irrespective of the choice of U .

Every great circle \mathcal{G} of the Bloch sphere (the state space of any two-level quantum mechanical system) can be identified with an element W of $SU(2)$ in the sense that $\langle\psi|W|\psi\rangle$ is same only for the elements $|\psi\rangle$ of \mathcal{G} . For example, the NOT operation $i\sigma_y$ brings every element of the polar great circle $\mathcal{G} = \{\cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle | \theta \in [0, \pi]\} \cup \{\cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle | \theta \in [0, \pi]\}$ to its orthogonal state but it does not do the same job for states lying outside of \mathcal{G} . The issue of finding out the geometric structure of state space for higher dimensional quantum systems is not yet resolved. So what would be the shape of the boundary⁸ of the intersection of this geometric structure and any

⁸The so called 'great circle' of the corresponding geometric structure of the state space.

hyperplane passing through the centre of this geometric body, is yet to be figured out, in general. Our present work may through some light on the structure of this boundary.

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